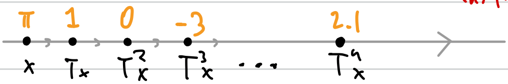
Measure Theory with Ergodic Horizons
Lecture 24

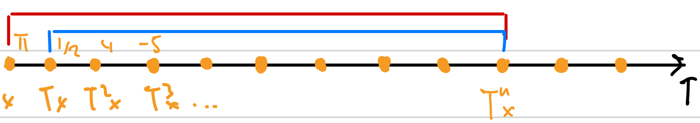
Classical Pointwise Ergodic Theorem (Birkhoff 1931). Let $(X,\mathcal{B},\mu)$ be a probability space. A $(\mathcal{B},\mathcal{B})$-measurable μ-preserving transformation T is ergodic iff for each $f\in L^1(X,\mu)$ and for a.e. $x\in X$,

$$\lim_n \underbrace{\left(\text{average } f \text{ over } \{x, Tx, T^2x, \dots, T^n x\}\right)}_{A_n f(x) := \frac{1}{n+1}\sum_{i=0}^{n} f\circ T^i(x)} = \int f\,d\mu.$$



Proof (⇒.) Invariance of limit, & tiling + local-global bridge.

Invariance of limit. Let $\bar{A}f := \limsup_{n\to\infty} A_n f$ and $\underline{A}f := \liminf_{n\to\infty} A_n f$. We claim that these functions are T-invariant, i.e. constant on each T-orbit. It is enough to verify that their values at every $x\in X$ are equal to their values at Tx. But for a fixed $x\in X$, this is immediate from the fact that $A_{n+1}f(x) \overset{(*)}{\approx_{\varepsilon_n}} \frac{n}{n+1} A_n f(Tx)$, where $\varepsilon_n := \frac{|f(x)|}{n+1}$.



Indeed, $\varepsilon_n \to 0$ and $\frac{n}{n+1} \to 1$, so $\limsup_{n\to\infty} A_{n+1}f(x) = \limsup_{n\to\infty} A_n f$, and same for liminf. $(*)$ follows from the following elementary lemma.

Lemma (about averages). Let X be a set and $f: X\to\mathbb{R}$. Let U, V be disjoint nonempty finite subsets of X. Let $A_U f :=$ the average of f over U, i.e. $\frac{1}{|U|}\sum_{x\in U} f(x)$. Then:

$$A_{U\sqcup V} f = \frac{|U|}{|U|+|V|} A_U f + \frac{|V|}{|U|+|V|} A_V f,$$ i.e. $A_{U\sqcup V} f$ is a convex combination of $A_U f$ and $A_V f$, weighted proportionally to U and V.

$U = \{x\}$

Proof. Direct verification. $\square$

Remark. This part is just the abstract statement that if $f: \mathbb{N} \to \mathbb{R}$, then
$$\limsup_{n\to\infty} A_{[0,n]} f = \limsup_{n\to\infty} A_{[1,n]} f.$$

T-invariance and the ergodicity of T imply (HW) that $\bar{A}f$ are $\underline{A}f$ are constant a.e., i.e. there are constants $\bar{c}$ and $\underline{c}$ such that the (T-invariant) sets $\bar{X} := \{x\in X : \bar{A}f(x) = \bar{c}\}$ and $\underline{X} := \{x\in X : \underline{A}f(x) = \underline{c}\}$ are conull. Thus, restricting to $\bar{X}\cap\underline{X}$, we may assume WLOG that $\bar{A}f \equiv \bar{c}$ and $\underline{A}f \equiv \underline{c}$.

&. By considering $f - \int f d\mu$ instead, we may assume WLOG that $\int f d\mu = 0$. To prove the theorem, we need to show that $\bar{A}f \le 0 \le \underline{A}f$. Suppose toward a contradiction, that $\bar{A}f \equiv \bar{c} > 0$, so $\bar{A}f > \Delta > 0$ for some $\Delta > 0$. (The case $0 < \underline{A}f$ is handled similarly.) Then let $x \mapsto n_x :=$ the least $n\in\mathbb{N}$ such that $A_{n_x} f(x) > \Delta > 0$.

We now make two assumptions which lose generality, and we will remove them later in HW:

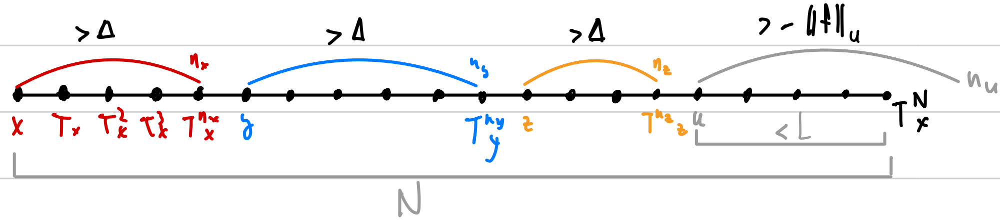
(i) f is bounded, i.e. $\|f\|_u := \sup_{x\in X} |f(x)| < \infty$.

(ii) $x \mapsto n_x$ is bounded, say by $L > 0$.

tiling. Note that if $x \mapsto n_x$ were constant, say 100, then this would immediately contradict the local-global bridge: $0 = \int f \, d\mu = \int A_{100} f(x) \, d\mu \geq \int \Delta \, d\mu \geq \Delta > 0$.

Thus okay $x \mapsto n_x$ is not constant, but letting $N \in \mathbb{N}$ be large enough so that $\frac{L}{N} \cdot \|f\|_u \leq \frac{\Delta}{3}$ and $\frac{L}{N} \leq \frac{1}{3}$, we get that $A_N f(x) \geq \frac{\Delta}{3}$ for all $x \in X$. Fix $x \in X$.

We tile the interval $I_N(x) := \{x, Tx, \dots, T^N x\}$ by intervals of the form $I_{n_y}(y)$, for which by the def. of n_y, we know that $A_{n_y} f(y) > \Delta$. We do this as follows:



Then we have tiled a subset $I \subseteq I_N(x)$ so that $|I_N(x) \setminus I| \leq \frac{L}{N} \cdot |I_N(x)|$. So $A_I f > \Delta$ and thus $A_N f(x) \geq \Delta/3$ because $|A_{I_N(x) \setminus I} f| \leq \|f\|_u$ (by the lemma about averages above).

$$A_N f(x) = \frac{|I|}{|I_N(x)|} \cdot A_I f + \frac{|I_N(x) \setminus I|}{|I_N(x)|} A_{I_N(x) \setminus I} f$$

$$\geq \left(1 - \frac{L}{N}\right) \cdot \Delta - \frac{L}{N} \|f\|_u \geq \frac{2}{3}\Delta - \frac{1}{3} \cdot \Delta = \frac{\Delta}{3}.$$

local-global bridge. But this contradicts the local-global bridge:

$$0 = \int f \, d\mu = \int A_N f \, d\mu \geq \int \frac{\Delta}{3} \, d\mu = \frac{\Delta}{3} > 0.$$

$\square$